§2 The canonical divisor

Recall: Cartier divisor $Ca(X) \twoheadrightarrow CaCl(X)$

Line bundles $Pic(X)$

Weil divisor $Z(X) \twoheadrightarrow Cl(X)$ under (1)

- $Ca(X) \twoheadrightarrow CaCl(X) \hookrightarrow Pic(X)$ almost/always "$\cong$" (e.g. for X projective or integral)

- $$\begin{array}{ccc} Ca(X) & \longrightarrow & Z(X) \\ \downarrow & & \downarrow \\ CaCl(X) & \longrightarrow & Cl(X) \end{array}$$
  - injective if X is normal
  - " "
  - isomorphism if X is factorial ©

- $U \subset X$ open, dense $\rightsquigarrow$ $Cl(X) \twoheadrightarrow Cl(U)$ restriction
  (The closure of a Weil divisor in U yields a Weil divisor in X.)

Remark: This does not work for Cartier divisors!
i.e. $CaCl(X) \longrightarrow CaCl(U)$ is not, in general, surj.

- If $\operatorname{codim}(X \setminus U) \geq 2$ $\rightsquigarrow$ $Cl(X) \cong Cl(U)$, $Z(X) \cong Z(U)$
  (see [Ha, II.6.5])

$\mathbb{Q}$-divisors: Elements in $Ca(X)_{\mathbb{Q}}$, $CaCl(X)_{\mathbb{Q}}$, $Z(X)_{\mathbb{Q}}$, $Cl(X)_{\mathbb{Q}}$ are called $\mathbb{Q}$-Cartier resp. $\mathbb{Q}$-Weil divisors

Still have
$$\begin{array}{ccc} Ca(X)_{\mathbb{Q}} & \longrightarrow & Z(X)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ CaCl(X)_{\mathbb{Q}} & \longrightarrow & Cl(X)_{\mathbb{Q}} \end{array}$$

Also: A Weil divisor $D \in Z(X)$ (or $Cl(X)$)
is called <u>$\mathbb{Q}$-Cartier</u> if the induced
class $D \in Z(X)_{\mathbb{Q}}$ (resp. $D \in Cl(X)_{\mathbb{Q}}$)
is in the image of
$$Ca(X)_{\mathbb{Q}} \to Z(X)_{\mathbb{Q}} \quad (\text{resp } CaCl(X)_{\mathbb{Q}} \to Cl(X)_{\mathbb{Q}})$$

Equivalently: $D \in Z(X)$ (or $Cl(X)$) is $\mathbb{Q}$-Cartier
if $\exists n \in \mathbb{Z}_{>0}$ s.t. $nD \in \operatorname{Im}(Ca(X) \to Z(X))$
(resp. $\in \operatorname{Im}(CaCl(X) \to Cl(X))$)

(Note: $D \in Z(X)$ $\mathbb{Q}$-Cartier iff $\overline{D} \in Cl(X)$ $\mathbb{Q}$-Cartier.)

<u>Definition</u>: X is called <u>$\mathbb{Q}$-factorial</u> if
$$CaCl(X)_{\mathbb{Q}} \twoheadrightarrow Cl(X)_{\mathbb{Q}}.$$

Clearly, factorial $\Rightarrow$ $\mathbb{Q}$-factorial.

<u>Remark</u>: • factorial $\Rightarrow$ normal, but a priori
$\mathbb{Q}$-factorial might not imply normal.
(Of course, for the definition of $Cl(X)$ and the
homom. $CaCl(X) \to Cl(X)$ one needs at least
regular in codimension one.)
We shall always assume that $\mathbb{Q}$-factorial includes normal.

• $f: Y \to X$ dominant
$$\rightsquigarrow f^*: CaCl(X)_{\mathbb{Q}} \to CaCl(Y)_{\mathbb{Q}}$$

<u>Warm-up</u>: Canonical divisor of a smooth variety $/k=\bar{k}$ X

Ω_X = cotangent sheaf $(= \Omega_{X/k})$

X smooth $\Leftrightarrow$ Ω_X locally free of rank $= \dim(X)$

$\rightsquigarrow$ $\omega_X := \det(\Omega_X) = \Lambda^d \Omega_X$, $d = \dim(X)$

$\in \mathrm{Pic}(X)$ <u>canonical line bundle</u>

Using $\begin{array}{ccc} \mathrm{Ca}(X) & \twoheadrightarrow & \mathrm{Pic}(X) \\ \downarrow & & \\ Z(X) & & \end{array}$: $\exists\, K_X \in Z(X)$ which is Cartier with $\mathcal{O}(K_X) = \omega_X$

$K_X \in Z(X)$ is called <u>the canonical divisor</u>

(Clearly, K_X is unique only up to linear equivalence!)
$\Rightarrow K_X \in Cl(X)$ is unique

<u>Rem</u>: If X projective, L ample, an explicit canonical divisor can be constructed as follows:

$0 \neq s_1 \in H^0(X, \omega_X \otimes L^n)$ $\quad n \gg 0$

$0 \neq s_2 \in H^0(X, L^n)$ $\quad n \gg 0$

$\rightsquigarrow$ $Z(s_1), Z(s_2) \in Z(X)$ zero divisors

$K_X := Z(s_1) - Z(s_2)$ is a canonical divisor.

In the following we shall assume: X normal variety $/k=\bar{k}$ (Condition (B) holds.)

<u>Facts</u>: • $\underline{X_{reg}} = \{x \in X \mid \mathcal{O}_{X,x} \text{ regular}\} \subset X$ open, dense subset. (For any variety X $/k=\bar{k}$)

$\rightarrow$ [Ha, II. 8.16]

$\rightsquigarrow$ $\underline{X_{sing}} := X \setminus X_{reg}$. proper, closed subset.

- X normal $\Rightarrow$ $\mathrm{codim}(X_{sing}) \geq 2$

<u>Def</u>. <u>The canonical divisor of a normal variety</u> $/k = \bar{k}$

$$K_X \in Z(X) \quad (\text{or } \in Cl(X)) \text{ such that}$$

$$K_X|_{X_{reg}} \in Z(X_{reg}) \quad (\text{resp. } \in Cl(X_{reg})) \text{ is the}$$

canonical divisor of the smooth variety X_{reg}

Since $Z(X) \simeq Z(X_{reg})$, $Cl(X) \simeq Cl(X_{reg})$ (for $\mathrm{codim}(X_{sing}) \geq 2$), is K_X uniquely determined up to linear equivalence.

<u>Remark</u>: For X normal, K_X is only defined as a Weil divisor!

<u>Def</u>. X is <u>Gorenstein</u> if K_X is Cartier, i.e. contained in the image of $Ca(X) \to Z(X)$ (resp. $CaCl(X) \to Cl(X)$).

X is <u>$\mathbb{Q}$-Gorenstein</u> if K_X is $\mathbb{Q}$-Cartier.

<u>Rem</u>: • Gorenstein and $\mathbb{Q}$-Gorenstein implicitly assume X normal, otherwise K_X would not even be defined.
• $\mathbb{Q}$-Gorenstein will be a frequent assumption, whenever one needs to pull-back the canonical divisor (which is not well-defined for Weil divisors).

The ramification formula

Let X, Y normal varieties $/ k = \bar{k}$, $f : Y \to X$ birational morphism. Assume f proper.

Aim: Compare K_X, K_Y

In order to define pull-back: X Gorenstein, i.e. K_X Cartier

Def. $\underline{Exc(f)} \subset Y$ is the closed subset

$$\{ y \in Y \mid f \text{ not a local iso. in } y \}$$

By definition $Exc(f) \subset Y$ closed.

Write $Exc(f) = E_1 \cup \ldots \cup E_n \cup Z$

$E_i \subset Y$ prime divisor, $\operatorname{codim}(Z) \geq 2$

Remarks: i) $Y \setminus Exc(f) \cong X \setminus f(Exc(f))$

(Indeed, by Zariski Main theorem ([Ha. III.11.4]) has f connected fibres. Hence, if $U \subset Y$ open with $f|_U \cong f(U)$, then $U = f^{-1} f(U)$.)

ii) Again by Zariski Main theorem,

$f^{-1}(x)$ is either one point or of positive dimension. Hence, $f : E_i \to X$ has positive dimensional fibres. Therefore, $\operatorname{codim} f(E_i) \geq 2$.

iii) If X is $\mathbb{Q}$-factorial, then $Exc(f)$ is purely of codimension one, i.e. $Z = \emptyset$.

(↗ [Debarre, p. 28])

As before $f: Y \to X$ proper, birational, X normal and Gorenstein. Consider $K_X \in Cl(X)$ as Cartier divisor. Then $f^*K_X \in CaCl(Y)$ is well-defined!

If $Exc(f) = E_1 \cup \ldots \cup E_n \cup Z$ as above

$\rightsquigarrow$ $f^*K_X|_{Y \setminus Exc(f)} = K_Y|_{Y \setminus Exc(f)} \in Cl(Y \setminus Exc(f))$

because $Y \setminus Exc(f) \cong X \setminus f(Exc)$

As $\operatorname{codim}(Z) \geq 2$, one has

$$Cl(Y \setminus Exc(f)) = Cl(Y \setminus \bigcup E_i)$$

$$(\quad Cl(Y \setminus Z) \quad = Cl(Y) \quad)$$

$$\bigoplus \mathbb{Z}[E_i] \hookrightarrow Cl(Y) \twoheadrightarrow Cl(Y \setminus \bigcup E_i) \qquad \text{exact}$$

because E_i are exceptional

$$\begin{array}{ccc} f^*K_X, K_Y & \longmapsto & f^*K_X|_{Y \setminus \bigcup E_i} = K_Y|_{Y \setminus \bigcup E_i} \\ \cap & & \\ Cl(Y) & \longrightarrow & Cl(Y \setminus \bigcup E_i) = Cl(Y \setminus Exc(f)) \end{array}$$

$\Rightarrow \exists!$ $a_1, \ldots a_n \in \mathbb{Z}$ unique

$$\boxed{K_Y = f^*K_X + \sum a_i [E_i]}$$ ramification formula

This equation has to be read in $Cl(Y)$ or in $Z(Y)$ modulo linear equivalence

(A priori K_Y and E_i might ~~be~~ not Cartier.)

Observation: In the above we have not really used that X is Gorenstein, i.e. K_X Cartier.

It would suffice to assume $\mathbb{Q}$-Gorenstein (and normal).

Then use $\bigoplus \mathbb{Q}[E_i] \hookrightarrow Cl(Y)_{\mathbb{Q}} \twoheadrightarrow Cl(Y \setminus \cup E_i)_{\mathbb{Q}}$
$$= Cl(Y \setminus \mathrm{Exc}(f))_{\mathbb{Q}}$$

$$K_Y = f^* K_X + \sum a_i [E_i] \qquad \underline{\underline{a_i \in \mathbb{Q}}}$$

Definition: If $f: Y \to X$ birational, proper

X $\mathbb{Q}$-Gorenstein and normal.

$$K_Y = f^* K_X + \sum a_i E_i.$$

a_i is called the discrepancy of E_i

$$\mathrm{discr}(X) = \inf \{ \mathrm{discr}(E) \mid E \text{ except. div. of some } Y \to X \text{ proper bir.} \}$$

(Arguments given below will show that one may or may not assume Y smooth.)

$k = \bar{k}$, char $k = 0$

__Def.__ A normal variety X has only __canonical__ (resp. __terminal__) singularities if

- X is $\mathbb{Q}$-Gorenstein, i.e. K_X is $\mathbb{Q}$-Cartier
- $\exists\, f : Y \to X$ proper, birational with Y smooth

  s.t. $K_Y = f^* K_X + \sum_{i=1}^{n} a_i E_i$ (E_i exceptional divisors)

  and $a_i \geq 0$ (respectively $a_i > 0$)

- Clearly, terminal $\Rightarrow$ canonical

- Smooth varieties have only terminal singularities.

__Proposition__: Both properties are independent of the resolution.

More precisely, if X is canonical (terminal) and $f' : Y' \to X$ is a resolution (i.e. f' proper, birational and Y' smooth), then

$$K_{Y'} = f'^* K_X + \sum_{i=1}^{n'} a_i' E_i' \quad \text{with } a_i' \geq 0 \ (\text{resp. } a_i' > 0),$$

where $E_1', \ldots, E_{n'}' \subset Y'$ are the exceptional divisors of f'.

__Proof__: We shall use the following three general results.

1) $f : Z \to X$, X, Z normal, f proper birational

   $F_1 \ldots F_k$ distinct prime divisors in Z, f-exceptional

   $\Rightarrow \bigoplus \mathbb{Z}[F_i] \subset Cl(Z)$

   (Already used several times.)

2) If $f : Y \to X$ is proper, birational, X, Y non-sing.

   $\Rightarrow K_Y = f^* K_X + \sum_{i=1}^{n} a_i E_i \quad a_i > 0$

   where $E_1, \ldots, E_n \subset Y$ are the (all!) exceptional divisors of f.

3) Suppose
$$\begin{tikzcd} Y \arrow[dr, "f"] & \\ & X \\ Y' \arrow[ur, "f'"'] & \end{tikzcd}$$
are two resolutions.

Then there exists a smooth variety $\tilde{Y}$ and proper, birational morphisms g, g' s.t.

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ \tilde{Y} \arrow[ur, "g"] \arrow[dr, "g'"'] \arrow[rr, phantom, "\circlearrowleft"] & & X \\ & Y' \arrow[ur, "f'"'] & \end{tikzcd}$$

This follows from resolution of singularities: Just desingularize $Y \times_X Y' \leftarrow \tilde{Y}$.

Using 1-3 one argues as follows:

By 3. it is enough to consider the situation

$$\begin{tikzcd} Y' \arrow[r, "g"] \arrow[rr, bend right, "f'"'] & Y \arrow[r, "f"] & X \end{tikzcd}$$
f, f' proper, birational

$E_1, \dots, E_n$ except. for f $\quad K_Y = f^*K_X + \sum a_i E_i$

$E'_1, \dots, E'_{n'}$ except for f' $\quad K_{Y'} = f'^*K_X + \sum a'_i E'_i$

Want: $\quad a_i \geq 0, i=1, \dots, n \iff a'_i \geq 0, \; i = 1, \dots, n'$

$\quad > 0 \qquad\qquad\qquad > 0$

Recall: $g: Z \to Y$ birational, proper and Y normal

If $E \subset Y$ divisor, then $\tilde{E}$ denotes the <u>proper</u> (or strict) <u>transform</u> of E which is the closure

$$\overline{E \setminus g(Exc(g))} \subset Z.$$

In our situation: $\{E'_j\} = \{\tilde{E}_i\} \cup \{F_i\}$

Wlog $\quad E'_1 = \tilde{E}_1, \dots, E'_n = \tilde{E}_n, E'_{n+1} = F_1, \dots, E'_{n'} = F_{n'-n}$

with F_i exceptional for g.

Then one obtains a ramification formula for $Y' \xrightarrow{g} Y$.

$$K_{Y'} = g^* K_Y + \sum_{j=1}^{k} b_j F_j \qquad k = n' - n$$

with $b_i > 0$ by 2)

and hence $$K_{Y'} = g^*\left(f^* K_X + \sum_{i=1}^{n} a_i E_i\right) + \sum_{j=1}^{k} b_j F_j$$
$$= f'^* K_X + \sum_{i=1}^{n} a_i g^* E_i + \sum_{j=1}^{k} b_i F_j.$$

(Note that $g^* E_i$ makes sense, as every divisor on the smooth variety Y is Cartier. Moreover,
$g^* E_i = \tilde{E}_i + \sum c_{ij} F_j$ with $c_{ij} \geq 0$.)

Hence:

$$K_{Y'} = f'^* K_X + \sum_{i=1}^{n} a_i \hat{E}_i + \sum_{j=1}^{k} \underbrace{\left(b_j + \sum_{i=1}^{n} c_{ij}\right)}_{>0 \text{ always}} F_j.$$

$$a_i' = a_i \qquad i = 1, \dots n$$

□

How to prove 2):

$f: Y \to X$ birational, X, Y smooth

$\Rightarrow$ $f^* \Omega_X \to \Omega_Y$ is generically an isomorphism and fits in a short exact sequence

$$0 \to f^* \Omega_X \to \Omega_Y \to \quad \to 0 .$$

Thus

$$0 \to f^* \omega_X \to \omega_Y \to \mathcal{F} \to 0 \quad \text{with}$$
$$\operatorname{supp}(\mathcal{F}) = \operatorname{Exc}(f) = \bigcup_{i=1}^{n} E_i$$

(Clearly, $\mathrm{Exc}(f) \supset \mathrm{supp}(\mathcal{F})$, but also $\mathrm{supp}(\mathcal{F}) \supset \mathrm{Exc}(f)$, as $\mathrm{Exc}(f)$ is purely of codimension one with fibers over X dimension ≥ 1.
Whenever $f^{-1}(x)$ in $y \in f^{-1}(x)$ has dimension ≥ 1, then $f^*\Omega_X \to \Omega_Y$ cannot be surjective in this point.)

The rest of the argument is very general:

$\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}(Y)$, $0 \to \mathcal{L} \to \mathcal{L}' \to \mathcal{F} \to 0$ exact with $\mathrm{supp}(\mathcal{F}) = \bigcup E_i$, $E_1, \dots E_n$ prime divisors

$$\Rightarrow \mathcal{L}' = \mathcal{L} \otimes \mathcal{O}\left(\sum a_i E_i\right) \quad a_i > 0$$

By induction $Y \setminus \bigcup^{n} E_i \subset Y \setminus \bigcup^{n-1} E_i \dots \subset Y \setminus E_1 \subset Y$

Wlog $n = 1$, $E = E_1$:

$0 \to \mathcal{O}(-E) \to \mathcal{O} \to \mathcal{O}_E \to 0$ yields exact sequence

$0 \to \mathcal{L}'(-E) \to \mathcal{L}' \to \mathcal{L}'_E \to 0$ and

$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_E \to 0$ $\qquad \mathcal{F}(-E) \twoheadrightarrow \mathcal{F}_1 = \mathrm{Ker}(\mathcal{F} \to \mathcal{F}_E)$

$$\begin{tikzcd}
 & & \mathcal{L}'(-E) \arrow[r] \arrow[d] & \mathcal{F}_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{L} \arrow[r] \arrow[ur, dotted] & \mathcal{L}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{L}'_E \arrow[r, two heads] \arrow[d] & \mathcal{F}_E \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

$\mathrm{supp}(\mathcal{F}) = E$, $\mathcal{L}'_E$ line bundle $\Rightarrow \mathcal{L}'_E \simeq \mathcal{F}_E$.

$\leadsto 0 \to \mathcal{L} \to \mathcal{L}'(-E) \to \mathcal{F}_1 \to 0$. Then either i) $\mathrm{supp}(\mathcal{F}_1) = E$ or ii) $\mathrm{codim}(\mathrm{supp}\, \mathcal{F}_1) \geq 2$.

i) go on, but process will stop somewhere.

ii) $\mathcal{L}'(-E) \simeq \mathcal{L}$ on $Y \setminus \mathrm{supp}(\mathcal{F}_1)$ and hence on Y.